Elliptic Integral Solutions to a Class of Space Flight Optimization Problems

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This paper is initially concerned with the minimum-time, exoatmospheric flight of a rocket with constant thrust acceleration magnitude, as in the cases of nuclear and solar electric propulsion. Gravitational acceleration is assumed to be a constant scalar multiple of the radius vector, plus a correction term which is a given function of time. The solution to the state equations is obtained in terms of elliptic integrals. A method is presented for the solution of the two-point boundary-condition problem associated with orbital transfer. At most, the latter method requires iteration upon final time, angle of injection, and two other parameters which are bounded. An example problem is provided which involves a rocket with very low thrust and a spiraling trajectory of many revolutions, but an altitude change of only several hundred miles above the earth. Finally, the original elliptic integral solution is extended to a larger class of low and intermediate thrust problems with constant thrust magnitude, mass decreasing with time, and an inverse square gravitational force. The extension is effected by deriving variation of parameters equations. The latter differential equations are somewhat analogous to equations of perturbed conic motion. However, all of the control time-histories depart from the optimal control for large changes in altitude.

Introduction

IN recent years there has been an interest in the use of rockets in the low and intermediate thrust range for the transfer of payloads from orbit to orbit about the earth or sun. Conceptual designs of a space tug for delivering payloads into geosyncronous orbit, for example, have included intermediate-thrust chemical rockets and low-thrust rockets with solar electric power. The trajectories of such rockets have powered phases varying from several degrees about the central body to many revolutions. The numerical integration of the equation of motion for these rockets can be very time-consuming.

The problem is first attacked in this paper by requiring the magnitude a of the thrust acceleration to be constant and the gravitational acceleration to be a constant scalar multiple of the radius vector plus a given function of time. Prior work with a similar model, in which a is not necessarily constant, is contained in Refs. 1 and 2.

Development of Equations

Consider the state equations

$$\ddot{\bar{R}} = a\hat{\lambda} - k^2\bar{R} + + \Delta\bar{G}(t) \tag{1}$$

where $\bar{R}$ is the radius vector in a sun- or earth-centered Cartesian coordinate system, $-k^2\bar{R}$ is an approximation to the gravitational acceleration, k^2 is a constant, a is the constant thrust acceleration magnitude, $\Delta\bar{G}(t)$ is a given function of time t, and $\hat{\lambda}$ is the optimum unit steering vector with $\bar{\lambda}$ being determined by the costate equations

$$\ddot{\bar{\lambda}} = -k^2\bar{\lambda} \tag{2}$$

Equations (2) are the Euler-Lagrange equations of optimal control theory which correspond to Eqs. (1). References 1 and 2 discuss Eqs. (1) and (2) with $\Delta\bar{G} = \bar{0}$ and $a = F/m$ where F is a constant and the mass m is a linear function of time. The solution to Eqs. (1) in such a problem cannot be expressed in terms of elliptical integrals. However, it can be expressed in terms of power series[1] or definite integrals.[2] The initial state $(t_f, \bar{R}_0, \bar{V})$ is given, and the conditions

$$g_j(t_f, \bar{R}_f, \bar{V}_f, \bar{\lambda}_f, \dot{\bar{\lambda}}_f) = 0 \quad (j = 1, \ldots, 6) \tag{3}$$

must be satisfied, where f signifies the final point and 0 the initial point. Here $\bar{V}$ is the velocity vector. Conditions for Eq. (3) may include transversality conditions as well as physical constraints. The six conditions for Eq. (3), along with the scaling condition $|\bar{\lambda}_0| = 1$ are usually sufficient to determine a solution $(t_p, \bar{\lambda}_0, \dot{\bar{\lambda}}_0)$.

Solution in Terms of Quadratures

The solution to Eq. (2) is simply

$$\bar{\lambda} = (\cos kt)\bar{\lambda}_0 + \frac{\sin kt}{k}\dot{\bar{\lambda}}_0 \tag{4}$$

The solution to Eq. (1) is

$$\bar{R} = \cos kt\left[\frac{1}{k}\int_0^t (\sin kt)(a\hat{\lambda} + \Delta\bar{G})dt + \bar{R}_0\right]$$

$$+ \sin kt\left[-\frac{1}{k}\int_0^t (\cos kt)(a\hat{\lambda} + \Delta\bar{G})dt + \frac{1}{k}\bar{V}_0\right] \tag{5a}$$

$$\bar{V} = \cos kt\left[\int_0^t (\cos kt)(a\hat{\lambda} + \Delta\bar{G})dt + \bar{V}_0\right]$$

$$+ \sin kt\left[\int_0^t (\sin kt)(a\hat{\lambda} + \Delta\bar{G})dt - k\bar{R}_0\right] \tag{5b}$$

assuming the initial value of time is zero. Substitution of the right-hand member of Eq. (4) for $\hat{\lambda}$ in Eqs. (5) yields

$$\bar{R} = \frac{\cos kt}{k}\left(-I_2\bar{\lambda}_0 - \frac{1}{k}I_3\dot{\bar{\lambda}}_0 - \bar{I}_5 + k\bar{R}_0\right)$$

$$+ \frac{\sin kt}{k}\left(I_1\bar{\lambda}_0 + \frac{1}{k}I_2\dot{\bar{\lambda}}_0 + \bar{I}_4 + \bar{V}_0\right) \tag{6a}$$

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$$\bar{V}=\cos kt(I_1\bar{\lambda}_0+\frac{1}{k}I_2\dot{\bar{\lambda}}_0+\bar{I}_4+\bar{V}_0)$$
$$+\sin kt(I_2\bar{\lambda}_0+\frac{1}{k}I_3\dot{\bar{\lambda}}_0+\bar{I}_5-k\bar{R}_0) \tag{6b}$$

where

$$I_1=\int_0^t\frac{a\cos^2 kt}{\lambda}dt,\quad I_2=\int_0^t\frac{a\cos kt\sin kt}{\lambda}dt,$$
$$I_3=\int_0^t\frac{a\sin^2 kt}{\lambda}dt,\quad I_4=\int_0^t(\cos kt)\Delta\bar{G}dt,$$
$$\bar{I}_5=\int_0^t(\sin kt)\Delta\bar{G}dt$$

and

$$\lambda^2=\lambda_0^2\cos^2 kt+2\bar{\lambda}_0\cdot\dot{\bar{\lambda}}_0\cos kt\cdot\sin kt/k+\dot{\lambda}_0^2\sin^2 kt/k^2$$

Here

$$\lambda=|\bar{\lambda}|,\ \lambda_0=|\bar{\lambda}(0)|,\text{ and }\dot{\lambda}_0=|\dot{\bar{\lambda}}_0|$$

Reduction to Elliptic Integrals

The variable t will be replaced with a new independent variable $T=kt+\tau$, where τ will be chosen such that the coefficient of $\cos T\sin T$, in the resulting expression for λ^2, will be identically zero. The result is

$$\tau=\frac{1}{2}\tan^{-1}\frac{-2k\bar{\lambda}_0\cdot\dot{\bar{\lambda}}_0}{k^2\lambda_0^2-\dot{\lambda}_0^2} \tag{7}$$

where 2τ is chosen in $[0,2\pi)$ according to the signs of the numerator and denominator of the argument of $\tan^{-1}$. Now

$$\lambda^2=\gamma^2\cos^2 T+\delta^2\sin^2 T \tag{8}$$

where

$$\gamma=\left|(\cos\tau)\bar{\lambda}_0-\frac{\sin\tau}{k}\dot{\bar{\lambda}}_0\right|,\quad \delta=\left|(\sin\tau)\bar{\lambda}_0+\frac{\cos\tau}{k}\dot{\bar{\lambda}}_0\right|$$

The problem considerably simplifies if $\gamma=0$, $\delta=0$, or $\gamma=\delta$. However, for the sake of brevity these special cases will not be considered further, and it will be assumed that $\gamma\neq0$, $\delta\neq0$, and $\gamma\neq\delta$.

It follows from Eq. (8) that

$$\lambda=\gamma(1-\theta^2\sin^2 T)^{1/2} \tag{9}$$

where $\theta=(\gamma^2-\delta^2)^{1/2}/\gamma$. It can be easily shown that $\gamma^2>\gamma^2-\delta^2>0$. Hence $\theta\in(0,1)$.

Letting $kt=T-\tau$ in the integrands of I_1, I_2, and I_3, it is possible to express the integrals as

$$\begin{bmatrix} I_1\\ I_2\\ I_3\end{bmatrix}=\frac{1}{2k\gamma}\begin{bmatrix}1+\cos2\tau & 2\sin2\tau & 1-\cos2\tau\\ -\sin2\tau & 2\cos2\tau & \sin2\tau\\ 1-\cos2\tau & -2\sin2\tau & 1+\cos2\tau\end{bmatrix}\times\begin{bmatrix}J_1\\ J_2\\ J_3\end{bmatrix} \tag{10}$$

where

$$J_1=\int_{T_1}^{T_2}\frac{a\cos^2 T}{(1-\theta^2\sin^2 T)^{1/2}}dt=\frac{a}{\theta^2}[E^*-(1-\theta^2)F^*]$$
$$J_2=\int_{T_1}^{T_2}\frac{a\cos T\sin T}{(1-\theta^2\sin^2 T)^{1/2}}dt=-\frac{a}{\theta^2}(1-\theta^2\sin^2 T)^{1/2}\Big|_{T_1}^{T_2}$$
$$J_3=\int_{T_1}^{T_2}\frac{a\sin^2 T}{(1-\theta^2\sin^2 T)^{1/2}}dt=\frac{a}{\theta^2}(F^*-E^*)$$

Here $T_1=\tau$, $T_2=\tau+kt$,

$$F^*(\theta,T_1,T_2)=\int_{T_1}^{T_2}(1-\theta^2\sin^2 T)^{-1/2}dt$$
$$E^*(\theta,T_1,T_2)=\int_{T_1}^{T_2}(1-\theta^2\sin^2 T)^{1/2}dt$$

Using Eqs. (10), Eqs. (6) may be expressed in terms of J_1,J_2 and J_3 as follows:

$$\bar{R}=(k\psi_1\bar{\lambda}_0-\psi_2\dot{\bar{\lambda}}_0)/(2k^3\gamma)+$$
$$[-(\bar{I}_5-k\bar{R}_0)\cos kt+(\bar{I}_4+\bar{V}_0)\sin kt]/k \tag{11a}$$
$$\bar{V}=(k\psi_3\bar{\lambda}_0+\psi_4\dot{\bar{\lambda}}_0)/(2k^2\gamma)+(\bar{I}_5-k\bar{R}_0)\sin kt$$
$$+(\bar{I}_4+\bar{V}_0)\cos kt \tag{11b}$$

where

$$\psi_1=(J_1+J_3)\sin kt+(J_1-J_3)\sin(2\tau+kt)$$
$$-2J_2\cos(2\tau+kt)$$
$$\psi_2=(J_1+J_3)\cos kt-(J_1-J_3)\cos(2\tau+kt)$$
$$-2J_2\sin(2\tau+kt)$$
$$\psi_3=(J_1+J_3)\cos kt+(J_1-J_3)\cos(2\tau+kt)$$
$$+2J_2\sin(2\tau+kt)$$
$$\psi_4=(J_1+J_3)\sin kt-(J_1-J_3)\sin(2\tau+kt)$$
$$+2J_2\cos(2\tau+kt)$$

Inversion of Eqs. (11) gives

$$\bar{\lambda}_0=\frac{k\gamma}{2(J_1J_3-J_2^2)}\{k\psi_4\bar{R}+\psi_2\bar{V}+\psi_5(\bar{I}_5-k\bar{R}_0)$$
$$+[-(J_1+J_3)+\psi_6](\bar{I}_4+\bar{V}_0)\} \tag{12a}$$
$$\dot{\bar{\lambda}}_0=\frac{k^2\gamma}{2(J_1J_3-J_2^2)}\{-k\psi_3\bar{R}+\psi_1\bar{V}-[(J_1+J_3)$$
$$+\psi_6](\bar{I}_5-k\bar{R}_0)+\psi_5(\bar{I}_4+\bar{V}_0)\} \tag{12b}$$

where

$$\psi_5=-(J_1-J_3)\sin2\tau+2J_2\cos2\tau$$
$$\psi_6=(J_1-J_3)\cos2\tau+J_2\sin2\tau$$

The Two-Point Boundary-Condition Problem

It will now be assumed that the problem is that of injection in minimum time into a specified orbit. Other problems can be treated similarly. Let ϕ be the range angle of the point of injection into the desired orbit. For any given value of ϕ, the final position and velocity vectors, $\bar{R}_f$ and $\bar{V}_f$, are uniquely determined.

Now the two-point boundary-condition problem can be solved in principle by iterating upon τ, θ, t_f, and ϕ as follows: Determine $\bar{R}_f$ and $\bar{V}_f$ from ϕ. Let $\lambda_0^2=1$ and solve the equations $\theta^2=(\gamma^2-\delta^2)/\gamma^2$ and $\tan\ 2\tau=-2k\bar{\lambda}_0\cdot\dot{\bar{\lambda}}_0/(k^2\lambda_0^2-\dot{\lambda}_0^2)$ for $\dot{\lambda}_0^2$ and $\bar{\lambda}_0\cdot\dot{\bar{\lambda}}_0$. The solution is

$$\dot{\lambda}_0{}^2=k^2\lambda_0^2\frac{1-\theta^2\cos^2\tau}{1-\theta^2\sin^2\tau} \tag{13}$$

$$\bar{\lambda}_0\cdot\dot{\bar{\lambda}}_0=-\frac{k}{2}\lambda_0{}^2\theta^2\frac{\sin^2\tau}{1-\theta^2\sin^2\tau} \tag{14}$$

Calculate γ^2 from τ and θ using Eqs. (13) and (14). It is given by the equation

$$\gamma^2=1/(1-\theta^2\sin^2\tau) \tag{15}$$

Calculate J_1, J_2, J_3, $\bar{I}_4$, and $\bar{I}_5$ from τ,θ, and t_f. Evaluate $\bar{\lambda}_0$ and $\dot{\bar{\lambda}}_0$ by means of Eq. (12), and calculate the corresponding values of the terms $\lambda_0^2, \bar{\lambda}_0\cdot\dot{\bar{\lambda}}_0$ and $\dot{\lambda}_0^2$. The values of τ, θ, t_f, and ϕ must be determined such that the latter values of λ_0^2, $\bar{\lambda}_0\cdot\dot{\bar{\lambda}}_0$, and $\dot{\lambda}_0^2$ are equal to the values obtained from the equations $\lambda_0^2=1$, (13) and (14), and such that a single transversality condition, associated with the orbital injection problem, is satisfied.

It will shown that the procedure described for solving the two-point boundary-condition problem can be considerably simplified in some low-thrust problems involving at least several revolutions about the central body. The greatest simplifications result from a restriction now placed upon the final time t_f; namely, the requirement that t_f satisfy the equation

$$kt_f=n\pi \tag{16}$$

where n is a positive integer to be chosen as small as possible, yet consistent with the necessary conditions of optimality. If there are at least several revolutions about the central body, then Eq. (16) gives $t_f \doteq t_{f\min}$. The procedure is to determine τ, θ, and ϕ such that the values of $\lambda_0{}^2=1$ and (13) and (14) are equal to the corresponding values obtained for $\bar{\lambda}_0$ and $\dot{\bar{\lambda}}_0$ as defined by Eqs. (12). The solution will be in terms of n which can be minimized. The resulting solution $(\bar{R}(t), \bar{V}(t), \bar{\lambda}(t), \dot{\bar{\lambda}}(t))$ will satisfy necessary conditions of optimality for a minimum time flight program carrying the rocket from $\bar{R}_0$, $\bar{V}_0$) to the final state $(\bar{R}_f, \bar{V}_f)$ corresponding to ϕ. The optimum flight path of this type should differ little from the optimum orbital transfer solution without the restriction imposed upon t_f by Eq. (16).

With $kt_f=n\tau$ one obtains $J_2=0$, $\sin kt_f=0$, $\cos kt_f=(-1)^n$, $\sin(2\tau+kt)=(-1)^n\sin2\tau$, $\cos(2\tau+kt)_f)=(-1)^n\cos2\tau$. Moreover, $F^*=2nK(\alpha)$ and $E^*=2nE(\alpha)$, where K and E are complete elliptic integrals of the first and second kinds and $\sin\alpha=\theta$. Then Eqs. (12) reduce to

$$\bar{\lambda}_0=\frac{k\gamma}{2J_1J_3}\{-k(J_1-J_3)(\sin2\tau)\bar{R}^*+[(J_1+J_3)-(J_1-J_3)\cos2\tau]\bar{V}^*\} \tag{17a}$$

$$\dot{\bar{\lambda}}_0=\frac{k^2\gamma}{2J_1J_3}\{-k[(J_1+J_3)+(J_1-J_3)\cos2\tau]\bar{R}^*+(J_1-J_3)(\sin2\tau)\bar{V}^*\} \tag{17b}$$

where

$$\bar{R}^*=(-1)^n\bar{R}_F-\bar{R}_0+\bar{I}_5/k,\quad \bar{V}^*=(-1)^n\bar{V}_F-V_0-\bar{I}_4$$

Forming expressions for $\lambda_0{}^2$, $\dot{\lambda}_0{}^2$, and $\bar{\lambda}_0\cdot\dot{\bar{\lambda}}_0$ from Eqs. (17) and substituting them into the equations $\lambda_0{}^2=1$,(13) and (14), one obtains

$$2L(1-\theta^2\sin^2\tau)=k^2M^2(\sin2\tau)^2|\bar{R}^*|^2-2kM(1-M\cos2\tau)\sin2\tau\bar{R}^*\cdot\bar{V}^*+(1-M\cos2\tau)^2|\bar{V}^*|^2 \tag{18a}$$

$$2L(1-\theta^2\cos^2\tau)=k^2(1+M\cos2\tau)^2|\bar{R}^*|^2-2kM(1+M\cos2\tau)(\sin2\tau)\bar{R}^*\cdot\bar{V}^*+M^2(\sin2\tau)^2|\bar{V}^*|^2 \tag{18b}$$

$$L\theta^2\sin2\tau=-k^2M(1+M\cos2\tau)\sin2\tau|\bar{R}^*|^2+k[M^2(2\sin^22\tau-1)+1]\bar{R}^*\cdot\bar{V}^*-M(1-M\cos2\tau)\sin2\tau|\bar{V}^*|^2 \tag{18c}$$

where $M=(J_1-J_2)/(J_1+J_3)$ and $L=(2/k^2)J_1{}^2J_3{}^2/(J_1+J_3)^2$. Equations (18) are three simultaneous equations in the three unknowns τ,θ, and ϕ. Considerable manipulation of Eqs. (18) yields

$$k^2|\bar{R}^*|^2+|\bar{V}^*|^2=(J_1{}^2+J_4{}^2)/k^2 \tag{19a}$$

$$k^3|\bar{R}^*\times\bar{V}^*|=J_1J_4 \tag{19b}$$

$$\tan2\tau=-2k\bar{R}^*\cdot\bar{V}^*/(k^2|\bar{R}^*|^2-|\bar{V}^*|^2) \tag{19c}$$

where $J_4=(1-\theta^2)^{1/2}J_3$. Observe that τ has been eliminated from the first two of Eqs. (19). Hence, the first two equations relate the unknowns, ϕ and θ, where ϕ appears in the left-hand members, and θ in the right-hand sides. The third equation relates τ and ϕ.

If the expressions for $\lambda_0{}^2$, $\dot{\lambda}_0{}^2$ and $\bar{\lambda}_0\cdot\dot{\bar{\lambda}}_0$ are found by means of Eqs. (17), and are substituted into Eq. (7), it can be seen that $\bar{\lambda}_0\cdot\dot{\bar{\lambda}}_0\ \geq\ 0$ if and only if $\bar{R}^*\cdot\bar{V}^*\geq 0$, and also that $k^2\lambda_0^2-\dot{\lambda}_0^2\geq 0$ if $k^2|\bar{R}^*|^2-|\bar{V}^*|^2\geq 0$. Therefore, employing the definition of the angle 2τ, the quadrant of 2τ can be determined from the third of Eqs. (19) and the signs of $-2k\bar{R}^*\cdot\bar{V}^*$ and $k^2|\bar{R}^*|^2-|\bar{V}^*|^2$. Next it will be shown that in some cases it is possible to eliminate ϕ from one of the first two of Eqs. (19).

Orbital Transfer between Circular Orbits

Consider an orbital transfer between circular orbits. Then $\bar{R}_0\cdot\bar{V}_0=0$ and $\bar{R}_f\cdot\bar{V}_f=0$. Choose the coordinate system in the plane of the second orbit with the x-axis along the intersection of the orbital planes (or chosen anywhere in the plane if the orbits are coplanar). Let ϕ be the angle between the x-axis and $\bar{R}_f$. Then $\bar{R}_f$ and $\bar{V}_f$ can be expressed in terms of ϕ. Select $\bar{R}_0$ along the x-axis and suppose $\bar{V}_f$ has the same sense as $\bar{V}_0$. Take $k^2=\mu/R_0^3$. Let $\Delta\bar{G}=0$. Hence $\bar{I}_4=\bar{I}_5=0$. Let $\hat{u}_1$ and $\hat{u}_2$ signify the x and y axes. Then $\hat{u}_1\cdot\bar{V}_0=0$, $\hat{u}_2\cdot\bar{R}_0=0$, $\hat{u}_1\cdot\bar{R}_0=R_0$, $\hat{u}_2\cdot\bar{V}_0=V_0\cos i$, $R_F=rR_0$, $V_0^2=\mu/R_0$, and $V_f^2=du/(rR_0)$, where $r=R_f/R_0$ and i is the angle between the orbital planes.

Now the first two of Eqs. (19) become

$$(\mu/R_0^2)^4\{(r\cos i+r^{-1/2})^2\cos^2\phi-2(-1)^n[(r^2+1)r^{-1/2}\cos i+r+1]\cos\phi+(r+1)(r^2+1)/r-(r\cos i-r^{-1/2})^2\}=J_1^2J_4^2 \tag{20a}$$

$$(r/R_0^2)^2[2(-1)^{n+1}(r^{-1/2}\cos i+r)\cos\phi + (r^3+2r+1)/r] = J_1^2+J_4^2 \tag{20b}$$

The first of the previous equations results when both members of the second of Eqs. (19) are squared and $|\bar{R}^*\times\bar{V}^*|^2$ is replaced by $|\bar{R}^*|^2|\bar{V}^*|^2-(\bar{R}^*\cdot\bar{V}^*)^2$. Clearly cos ϕ can be eliminated from Eqs. (20).

Coplanar Transfer between Circular Orbits

When $i=0$, Eqs. (20) reduce to

$$(\mu/R_0^2)^4[(r+r^{-1/2})\cos\phi-(-1)^n(r^{1/2}+1)]^2 = J_1^2J_4^2 \tag{21a}$$

$$(\mu/R_0^2)^2[2(-1)^{n+1}(r+r^{-1/2})\cos\phi+(r^3+2r+1)/r] = J_1^2+J_4^2 \tag{21b}$$

Elimination of cos ϕ from Eqs. (21) gives

$$(\mu/R_0^2)\,|r-r^{-1/2}| = |J_1\pm(-1)^nJ_4| \tag{22}$$

Using the definitions of J_1, J_4, E^*, and F^*, one can show that

$$|J_1\pm(-1)^nJ_4| = 2na\frac{E(\alpha)+(-1)^nK(\alpha)\cos\alpha}{1+(-1)^n\cos\alpha} \tag{23}$$

where $\sin\alpha=\theta$ and $\alpha\epsilon(0,\pi)$. The plus sign applies if $\alpha\epsilon(0,\pi/2]$, and the minus applies if $\alpha\epsilon(\pi/2,\pi)$. Define a parameter α' such that $\alpha'=\alpha$ if n is even and $\alpha'=\pi-\alpha$ if n is odd. Then $\cos\alpha'=(-1)^n\cos\alpha$ and $\alpha'\epsilon(0,\pi)$. Observe that $E(\alpha)=E(\alpha')$ and $K(\alpha)=K(\alpha')$ because $\sin\alpha=\sin\alpha'$. Therefore, Eq. (23) may be written as

$$|J_1\pm(-1)^nJ_4| = 2naq(\alpha') \tag{24}$$

where $q(\alpha')=[E(\alpha')+K(\alpha')\cos\alpha']/(1+\cos\alpha')$. Combining Eqs. (22) and (24), one obtains

$$q(\alpha')=\frac{\mu}{2naR_0^2}\frac{|r^{3/2}-1|}{r^{1/2}} \tag{25}$$

where $q(\alpha')$ ranges from zero to $\pi/2$. Let $q^*(\alpha')=[E(\alpha')-K(\alpha')\cos\alpha']/(1-\cos\alpha')$ and $\beta(\alpha')=q^*(\alpha')/q(\alpha')$.

Equation (25) can be used to determine α' once n has been selected. For a given value of n there can be at most one solution to Eq. (25) because $q(\alpha')$ is monotonic. Thus only one sign in Eq. (22) will apply.

Combining Eqs. (21), substituting $2naq^*(\alpha')$ for $|J_1\mp(-1)^nJ_4|$, and employing Eq. (25), one obtains

$$\cos\phi=\frac{(-1)^n}{4(r^2+r^{1/2})}[r^3+2r^{3/2}+4r+1-(r^{3/2}-1)^2\beta^2(\alpha')] \tag{26}$$

Equation (26) implies that α' must be chosen such that

$$|r^3+2r^{3/2}+4r+1-(r^{3/2}-1)^2\beta^2(\alpha')|\leq 4(r^2+r^{1/2}) \tag{27}$$

The smallest positive integer n is sought such that

$$\alpha'=q^{-1}(A/n) \tag{28}$$

exists and satisfies inequality (27), where $A=\mu\;|r^{3/2}-1|/(2aR_0^2r^{1/2})$. Equation (28) is a restatement of Eq. (25). The smallest n is desired in order that $t_F=n\pi/k$ will be a minimum.

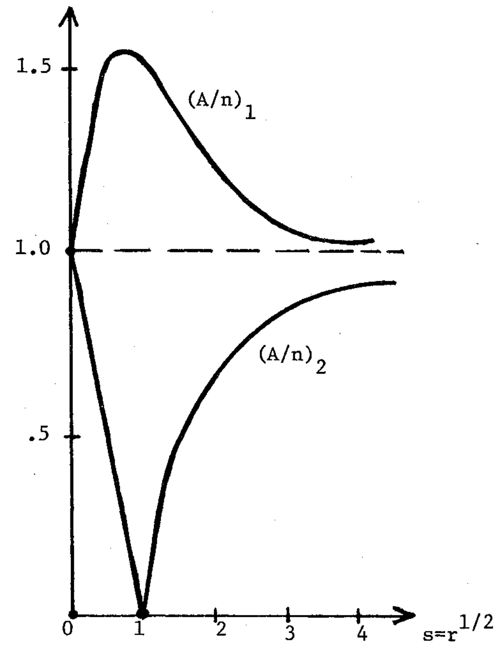

Fig. 1 (A/n_1) and $(A/n)_2$ vs s.

Inequality (27) may be stated in the equivalent form $b_1\leq\beta(\alpha')\leq b_2$ where $s=r^{1/2}$, $b_1=|s^2+s-1|(s^2+s+1)$, and $b_2=(s^3+2s+1)/[|s-1|(s^2+s+1)]$. Using Eq. (28), it is clear that n must be chosen such that $b_1\leq\beta[q^{-1}(A/n)]\leq b_2$. It can be shown that if $r>0$, then $b_1<1$ and $b_2>1$. Let $(A/n)_i$ be the value of A/n such that $\beta\{q^{-1}[(A/n)_i]\}=b_i(s)$ for $i=1,2$. In Fig. 1, $(A/n)_1$ and $(A/n)_2$ are plotted as functions of s. For a given value of s (or r), one seeks the smallest positive integer n such that $(A/n)_2\leq A/n\leq(A/n)_1$.

Consider $r\epsilon[1/4,4]$. Then $s\epsilon[1/2,2]$ in which case $(A/n)_1>1.2$ and $(A/n)_2<0.7$. Therefore, any integer n such that $0.7\leq A/n\leq 1.2$ will satisfy inequality (27).

Consider the optimal transfer of a low-thrust rocket from a circular orbit of 100 nautical miles altitude to a coplanar circular orbit of altitude equal to about 500 nautical miles. Take $\mu=0.3981\times10^{15}$ m^3/sec^2, $a=0.033806$ m/sec^2, $R_0=6.57\times10^6$ m, $r=1.0410$. Then $A=16.791$ and $s=1.0410$. From Fig. 1 it is evident that one must select the smallest n such that $A/n\leq1.48$. Therefore $n=12$ and $A/n\doteq1.3992$ in which case $\alpha'=\alpha\doteq77°$ because $q(77°)\doteq A/n$. Then $\theta^2=\sin^2\alpha\doteq0.94940$ and $\cos\phi\doteq0.99998$ as calculated from Eq. (26). Hence $\phi\doteq\pm0.38°$. Either sign may be selected. Taking the plus sign, one obtains $\tau\doteq98.325°$, $\bar{R}_f^T\doteq$ [7119.8 km, 47.2 km], $\bar{V}_f^T\doteq$ [−49.6 m/sec, 7477.4 m/sec], $\bar{\lambda}_0^T\doteq[-0.41197, 0.91144]$, $\dot{\bar{\lambda}}_0^T\doteq[-0.0043852, -0.005244]$.

Problems involving little altitude change and many revolutions about the central body are most likely to involve very low thrust rockets. Since the specific impulse is so high in such cases, there is little loss of mass. Therefore, the requirement of continuous thrusting (as opposed to bang-bang control) is not unreasonable.

Variation of Parameters

The method of variation of parameters[3] will be used to derive differential equations which can be integrated numerically to obtain a solution to the equation

$$\ddot{\bar{R}}=(F/m)\hat{\lambda}-(\mu/R^3)\bar{R}+\Delta\bar{G}(t) \tag{29}$$

where $m=m_0-\beta t$, $\ddot{\bar{\lambda}}=-k^2\bar{\lambda}$, and β and k are constants (over subarcs). Clearly the costate equations are based upon a simplified gravity model. The variation of parameters equations will be more amenable to numerical solution than Eq. (29) provided altitude and mass do not change rapidly. Moreover, if the overall change in altitude is sufficiently small (say several hundred miles or less), then the solution will be nearly optimal; in fact, it has been demonstrated[4] that —in many practical problems—nearly optimal trajectories can be obtained under the assumption that $\ddot{\lambda}=0$.

In order to derive the variation of parameters equations, (29) is first expressed as follows: $\dot{\bar{R}}=\bar{V}$ and $\dot{\bar{V}}=[a\hat{\lambda}-k^2\bar{R}+\Delta\bar{G}(t)]+[(F/m-a)\hat{\lambda}-(\mu/R^2-k^2)\bar{R}]$. Thus $\dot{\bar{V}}=f+g$, where an elliptic integral solution has been obtained for the equations, $\dot{\bar{R}}=\bar{V}$ and $\dot{\bar{V}}=f$. For given values of $\hat{\lambda}_0$ and $\dot{\hat{\lambda}}_0$, the elliptic integral solution may be expressed as $\bar{V}=\bar{\psi}(t,\bar{V}_0,\bar{R}_0)$ and $\bar{R}=\bar{\phi}(t,\bar{V}_0,\bar{R}_0)$. Identify $\bar{Y}$ with $\bar{V}_0$ and $\bar{Z}$ with $\bar{R}_0$. Then $\bar{V}=\bar{\psi}(t,\bar{Y},\bar{Z})$ and $\bar{R}=\bar{\phi}(t,\bar{Y},\bar{Z})$. The variation of parameters equations[3] are obtained by solving the equations

$$\frac{\partial\bar{\phi}}{\partial\bar{Y}}\dot{\bar{Y}}+\frac{\partial\bar{\phi}}{\partial\bar{Z}}\dot{\bar{Z}}=\bar{0}$$

$$\frac{\partial\bar{\psi}}{\partial\bar{Y}}\dot{\bar{Y}}+\frac{\partial\bar{\psi}}{\partial\bar{Z}}\dot{\bar{Z}}=\left(\frac{F}{m}-a\right)\hat{\lambda}-\left(\frac{\mu}{\phi^3}-k^2\right)\bar{\phi}$$

for $\dot{\bar{Y}}$ and $\dot{\bar{Z}}$. On doing this one obtains the differential equations

$$\dot{\bar{Y}}=\cos kt\left[\frac{a\beta t}{m}\hat{\lambda}-k^2\left(\frac{R_a^3}{\phi^3}-1\right)\bar{\phi}(t,\bar{Y},\bar{Z})\right] \tag{30a}$$

$$\dot{\bar{Z}}=-\frac{\sin kt}{k}\left[\frac{a\beta t}{m}\hat{\lambda}-k^2\left(\frac{R_a^3}{\phi^3}-1\right)\bar{\phi}(t,\bar{Y},\bar{Z})\right] \tag{30b}$$

where $k^2=\mu/R_a^3$ and R_a is some constant value of R. The solution $(\bar{V},\bar{R})$ to Eq. (29) is given by $\bar{V}=\bar{\psi}(t,\bar{Y},\bar{Z})$ and $\bar{R}=\bar{\phi}(t,\bar{Y},\bar{Z})$.

As a simple example of the use of Eqs. (30), consider a problem with $F=0$ and $\Delta\bar{G}=0$ in which a space vehicle descends from 270 to 65 nautical miles above the earth through a range angle of about 120°. Both Eqs. (29) and (30) were integrated in five steps with a fourth-order Runge-Kutta formula. Equations (30) gave an error in R_f of about 3000 ft, compared to an error of about 20,000 ft in the case of Eq. (29).

References

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